

# Generalized Super Bell Polynomials with Applications to Supersymmetric Equations

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**Abstract.** In this paper, we introduce a class of new generalized super Bell polynomials on a superspace, explore their properties, and show that they are a natural and effective tool to systematically investigate integrability of supersymmetric equations. The connections between the super Bell polynomials and super bilinear representation, bilinear Bäcklund transformation, Lax pair and infinite conservation laws of supersymmetric equations are established. We take supersymmetric KdV equation and supersymmetric sine-Gordon equation to illustrate this procedure.

**Keywords:** super Bell polynomial; supersymmetric equation; bilinear Bäcklund transformation; Lax pair; conservation law.

## 1. Introduction

The supersymmetry represents a kind of symmetrical characteristic between boson and fermion in physics. The concept of supersymmetry was originally introduced and developed for applications in elementary particle physics thirty years ago [1]–[3]. It is found that supersymmetry can be applied to a variety

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of problems such as relativistic, non-relativistic physics and nuclear physics. In recent years, supersymmetry has been a subject of considerable interest both in physics and mathematics. The mathematical formulation of the supersymmetry is based on the introduction of Grassmann variables along with the standard ones [36]. In a such way, a number of well known mathematical physical equations have been generalized into the supersymmetric analogues, such as supersymmetric versions of sine-Gordon, KdV, KP hierarchy, Boussinesq, MKdV etc. It has been shown that these supersymmetric integrable systems possess bi-Hamiltonian structure, Painlevé property, infinite many symmetries, Darboux transformation, Bäcklund transformation, bilinear form, super soliton solutions and super quasi-periodic solutions [4]–[18]. In our present paper, we investigate the integrability of supersymmetric equations by using a class of super Bell polynomials which are a multidimensional and super generalization of ordinary Bell polynomials.

The ordinary Bell polynomials introduced by Bell during the early 1930s are a class of exponential polynomials, which are specified by a generating function and exhibit important properties [22]. The Bell polynomials have been exploited in combinatorics, statistics and other fields [23]–[25]. Some generalized forms of Bell polynomials already appeared in literature [26]–[30]. More recently Lambert, Gilson et al found that the Bell polynomials also play important role in the characterization of bilinearizable equations. They presented an alternative procedure based on the use of the properties of Bell polynomials to obtain parameter families of bilinear Bäcklund transformation for soliton equations. As a

consequence bilinear Bäcklund transformation with single field can be linearize into corresponding Lax pairs [31]-[33].

Our paper is a further contribution to the theory of Bell polynomials and supersymmetric equations. We reconsider Bell polynomials in a more extended context— superspace. We define a kind of new generalized super Bell polynomials and discuss their relations with super bilinear equations, which actually provides an approach to systematically investigate complete integrability of supersymmetric systems. As illustrative examples, the bilinear representations, bilinear Bäcklund transformations, Lax pairs and infinite conservation laws of the supersymmetric KdV equation and supersymmetric sine-Gordon equation are obtained in a quick and natural manner.

The layout of this paper is as follows. In Section 2, we briefly recall elementary notations about superdifferential, integrals and super bilinear operators on superspace. As In Section 3, we propose theory of super Bell polynomials and establish their connections with supersymmetric equations. As consequence a approach to investigate integrability of supersymmetric equations is presented. In the Sections 4 and 5, as applications of super Bell polynomials, we study integrability of supersymmetric KdV equation supersymmetric sine-Gordon equation, respectively. At last, we briefly discuss further possible generalization and applications of Bell polynomials and future work in Section 6.

## 2. Derivatives and bilinear operators on superspace

To make our presentation easily understanding and self-contained, in this section we first briefly review some notations about superanalysis [34]-[37] and super-Hirota bilinear operators [19, 20].

A superalgebra is a  $Z_2$ -graded space  $\Lambda = \Lambda_0 \oplus \Lambda_1$  in which,  $\Lambda_0$  is a subspace consisting of even elements and  $\Lambda_1$  is a subspace consisting of odd elements. A parity function is introduced for homogeneous elements on the  $\Lambda$ , namely,  $|a| = 0$  if  $a \in \Lambda_0$  and  $|a| = 1$  if  $a \in \Lambda_1$ .

The superalgebra is said to be commutative if the supercommutator  $[a, b] = ab - (-1)^{|a||b|}ba = 0$ , for arbitrary homogeneous elements  $a, b \in \Lambda$ .

A commutative superalgebra  $\Lambda$  with unit  $e = 1$  is called a finite-dimensional Grassmann algebra if it contains a system of anticommuting generators  $\theta_j, j = 1, \dots, n$  with the anticommutative property:  $[\theta_j, \theta_k] = \theta_j\theta_k + \theta_k\theta_j = 0$ ,  $\theta_j^2 = 0$ ,  $j, k = 1, 2, \dots, n$ .

Let  $\Lambda = \Lambda_0 \oplus \Lambda_1$  be a finite-dimensional Grassmann algebra, then the Banach space  $\mathbb{R}_\Lambda^{m,n} = \Lambda_0^m \times \Lambda_1^n$  is called a superspace of dimension  $(m, n)$  over  $\Lambda$ . In particular, if  $\Lambda_0 = \mathbb{C}$  and  $\Lambda_1 = 0$ , then  $\mathbb{R}_\Lambda^{m,n} = \mathbb{C}^m$ . We may take even-valued complex space  $\Lambda_0 = \mathbb{C}$  in our context.

A function  $f(\mathbf{x}, \boldsymbol{\theta}) : \mathbb{R}_\Lambda^{m,n} \rightarrow \Lambda$  is said to be superdifferentiable at the point  $(\mathbf{x}, \boldsymbol{\theta}) \in \mathbb{R}_\Lambda^{m,n}$  with even coordinates  $\mathbf{x} = (x_1, \dots, x_m)$  and odd coordinates  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ , if there exist elements  $F_j(\mathbf{x}, \boldsymbol{\theta}), \tilde{F}_k(\mathbf{x}, \boldsymbol{\theta}) \in \Lambda$ ,  $j = 1, \dots, m; k = 1, \dots, n$ , such that

$$f(\mathbf{x} + \mathbf{h}, \boldsymbol{\theta} + \tilde{\mathbf{h}}) = f(\mathbf{x}, \boldsymbol{\theta}) + \sum_{j=1}^m \langle F_j(\mathbf{x}, \boldsymbol{\theta}), h_j \rangle + \sum_{k=1}^n \langle \tilde{F}_k(\mathbf{x}, \boldsymbol{\theta}), \tilde{h}_k \rangle + o(\|(\mathbf{h}, \tilde{\mathbf{h}})\|),$$

where the vectors  $\mathbf{h} = (h_1, \dots, h_m) \in \Lambda_0^m$  and  $\tilde{\mathbf{h}} = (\tilde{h}_1, \dots, \tilde{h}_n) \in \Lambda_1^n$ . The  $F_j(\mathbf{x}, \boldsymbol{\theta}), \tilde{F}_k(\mathbf{x}, \boldsymbol{\theta})$  are called the super partial derivative of  $f$  with respect to  $x_j, \theta_k$  at the point  $(\mathbf{x}, \boldsymbol{\theta})$  and are denoted, respectively, by

$$\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial x_j} = F_j(\mathbf{x}, \boldsymbol{\theta}), \quad \frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \theta_k} = F_k(\mathbf{x}, \boldsymbol{\theta}), \quad j = 1, \dots, m; k = 1, \dots, n.$$

The derivatives  $\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial x_j}$  with respect to even variables  $x_j, j = 1, 2, \dots, m$  are uniquely defined. While the derivatives  $\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \theta_k}$  to odd variables  $\theta_k, k = 1, 2, \dots, n$  are not uniquely defined, but with an accuracy to within an addition constant  $c\theta_1 \cdots \theta_n, c \in \Lambda_0$  from the annihilator  ${}^\perp L_n, L_n = \{\theta_1 \cdots \theta_n, \boldsymbol{\theta} \in \Lambda_1^n\}$ .

Let  $f = f(\mathbf{x}, \boldsymbol{\theta}), g = g(\mathbf{x}, \boldsymbol{\theta}) : \mathbb{R}_\Lambda^{m,n} \rightarrow \Lambda$  be a superdifferentiable function, then super derivative also satisfies Leibnitz formula

$$\partial_{x_j}(fg) = (\partial_{x_j}f)g + f(\partial_{x_j}g), \quad j = 1, \dots, m,$$

$$\partial_{\theta_k}(fg) = (\partial_{\theta_k}f)g + (-1)^{|f|}f(\partial_{\theta_k}g), \quad k = 1, \dots, n.$$

Let differential operators  $\mathcal{D}_k = \partial_{\theta_k} + \theta_k \partial_{x_r}$  ( $k = 1, \dots, n; r \in \{1, \dots, m\}$ )

be supersymmetric covariant derivatives, we can show that they satisfy

$$\begin{aligned} \mathcal{D}_k(fg) &= (\mathcal{D}_k f)g + (-1)^{|f|}f(\mathcal{D}_k g), \\ [\mathcal{D}_j, \mathcal{D}_k] &= 0, \quad \mathcal{D}_k^2 = \partial_{x_r}. \end{aligned} \tag{2.1}$$

Denote by  $\mathcal{P}(\Lambda_1^n, \Lambda)$  the set of polynomials defined on  $\Lambda_1^n$  with value in  $\Lambda$ .

We say that a super integral is a map  $I : \mathcal{P}(\Lambda_1^n, \Lambda) \rightarrow \Lambda$  satisfying the following condition is an super Berezin integral about Grassmann variables

(1) A linearity:  $I(\mu f + \nu g) = \mu I(f) + \nu I(g), \mu, \nu \in \Lambda, f, g \in \mathcal{P}(\Lambda_1^n, \Lambda);$

(2) translation invariance:  $I(f_\xi) = I(f)$ , where  $f_\xi = f(\boldsymbol{\theta} + \boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \Lambda_1^n, f \in \mathcal{P}(\Lambda_1^n, \Lambda)$ .

We denote  $I(\theta^\varepsilon) = I_\varepsilon$ , where  $\varepsilon$  belongs to the set of multiindices  $N_n = \{\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n), \varepsilon_j = 0, 1, \boldsymbol{\theta}^\varepsilon = \theta_1^{\varepsilon_1} \cdots \theta_n^{\varepsilon_n} \neq 0\}$ . In the case when  $I_\varepsilon = 0, \varepsilon \in N_n, |\varepsilon| \leq$

$n = n - 1$ , such kind of integral has the form

$$I(f) = J(f)I(1, \dots, 1) \equiv \frac{\partial^n f(0)}{\partial \theta_1 \dots \partial \theta_n} I(1, \dots, 1),$$

Since the derivative is defined with an accuracy to with an additive constant form the annihilator  ${}^\perp L_n$ , it follows that  $J : \mathcal{P} \rightarrow \Lambda / {}^\perp L_n$  is single-valued mapping. This mapping also satisfies the conditions 1 and 2, and therefore we shall call it an integral and denote

$$J(f) = \int f(\boldsymbol{\theta}) d\boldsymbol{\theta} = \int \theta_1 \dots \theta_n d\theta_1 \dots d\theta_n,$$

which has properties:

$$\begin{aligned} \int \theta_1 \dots \theta_n d\theta_1 \dots d\theta_n &= 1, \\ \int \frac{\partial f}{\partial \theta_j} d\theta_1 \dots d\theta_n &= 0, \quad j = 1, \dots, n. \\ \int f(\boldsymbol{\theta}) \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_j} d\boldsymbol{\theta} &= (-1)^{1+|g|} \int \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_j} g(\boldsymbol{\theta}) d\boldsymbol{\theta}. \end{aligned} \tag{2.2}$$

For a pair of Grassmann-valued functions  $f(\mathbf{x}, \boldsymbol{\theta})$ ,  $g(\mathbf{x}, \boldsymbol{\theta}) : \mathbb{R}_\Lambda^{m,n} \rightarrow \Lambda$ , the ordinary Hirota bilinear operator is defined by

$$D_{x_j} f(\mathbf{x}, \boldsymbol{\theta}) \cdot g(\mathbf{x}, \boldsymbol{\theta}) = (\partial_{x_j} - \partial_{x'_j}) f(\mathbf{x}, \boldsymbol{\theta}) g(\mathbf{x}', \boldsymbol{\theta}')|_{\mathbf{x}'=\mathbf{x}, \boldsymbol{\theta}'=\boldsymbol{\theta}}, \quad j = 1, \dots, m,$$

and super-Hirota bilinear operators are defined as

$$S_k D_{x_j} f(\mathbf{x}, \boldsymbol{\theta}) \cdot g(\mathbf{x}, \boldsymbol{\theta}) = (\mathcal{D}_k - \mathcal{D}'_k) f(\mathbf{x}, \boldsymbol{\theta}) g(\mathbf{x}', \boldsymbol{\theta}')|_{\mathbf{x}'=\mathbf{x}, \boldsymbol{\theta}'=\boldsymbol{\theta}}, \quad k = 1, \dots, n,$$

here we have denoted  $\mathcal{D}'_k = \partial_{\theta'_k} + \theta'_k \partial_{x_k}$ .

It can be shown that these super-Hirota bilinear operators have properties

$$S_k^{2N} f \cdot g = D_{x_r}^N f \cdot g, \quad N \in \mathbb{Z}$$

$$S_k f \cdot g = (\mathcal{D}_k f) g - (-1)^{|f|} f (\mathcal{D}_k g).$$

In our context, we are interested in bosonic, also called even, superfield function  $f(\mathbf{x}, \boldsymbol{\theta}) : \mathbb{R}_\Lambda^{m,n} \rightarrow \mathbb{R}_\Lambda^{1,0} = \Lambda_0$ . It can be expanded in powers of odd coordinates

$\theta_k$ ,  $k = 1, \dots, n$ , that is,

$$f = f_0(\mathbf{x}) + \sum_{k \geq 0} \sum_{j_1 < \dots < j_k} f_{j_1 \dots j_k}(\mathbf{x}) \theta_{j_1} \dots \theta_{j_k},$$

where the coefficients  $f_{j_1 \dots j_k}(\mathbf{x}) \in \Lambda_0$  are even functions with respect to  $x_1, \dots, x_m$ .

### 3. Generalized super Bell polynomials on superspace

Based on the above fundamental notations, in this section we develop theory of generalized super Bell polynomials, which are a main tool to study the integrability of supersymmetric equations.

#### 2.1. Generalized super Bell polynomials

To well compare our super Bell polynomials with ordinary ones, let's first simply recall the ordinary Bell polynomials. During the early 1930s, Bell introduced three kinds of exponential polynomials [22].

The first Bell polynomials are defined as

$$\xi_n(x, t, r) = e^{-tx^r} \partial_x^n e^{tx^r}, \quad (3.1)$$

where  $r > 0$  is a constant integer,  $n \geq 0$  an arbitrary integer, and  $x, t \in \mathbb{R}$  independent variables. For  $r = 2$ , the Bell polynomials  $\xi_n(x, t)$  are exactly Hermite polynomials.

The Bell polynomials are algebraic polynomials in two elements  $x$  and  $t$ . The first few lowest order Bell Polynomials are

$$\xi_0(x, t, r) = 1, \quad \xi_1(x, t, r) = rtx^{r-1}, \quad \xi_2(x, t, r) = r^2 t^2 x^{2r-2} + r(r-1)tx^{r-2},$$

$$\xi_3(x, t, r) = r^3 t^3 x^{3r-3} + 3r^2(r-1)t^2 x^{2r-3} + r(r-1)(r-2)tx^{r-3}.$$

The second Bell polynomials are a generalization of the Bell polynomials (3.1) and defined by

$$\phi_n = \phi(\alpha_1, \dots, \alpha_n), \quad \phi_0 = 1, \quad \phi_{n+1} = \sum_{s=1}^n \binom{n}{s} \alpha_{s+1} \phi_{n-s},$$

where  $(\alpha_1, \dots, \alpha_n, \dots)$  is an infinite sequence of independent variables. For the particular sequence  $\alpha_j = j! \binom{\ell_i}{r_i} x t^{r-j}$ ,  $j = 1, \dots, r$ ;  $\alpha_j = 0$ ,  $j > r$ , we have

$$\phi_n = \xi_n(x, t, r).$$

The Bell polynomial  $\phi_n$  is a polynomial about variables  $\alpha_1, \dots, \alpha_n$ . For instance, the first three of second Bell polynomials read

$$\phi_0 = 1, \quad \phi_1 = \alpha_1^2 + \alpha_2, \quad \phi_3 = \alpha_1^3 + 3\alpha_1\alpha_2 + \alpha_3.$$

The third Bell polynomials, further generalization of the  $\xi_n$  and  $\phi_n$ , are defined by

$$Y_n = Y_n(y_t, \dots, y_{nt}) = e^{-y} \partial_t^n e^y, \quad (3.2)$$

where  $y = e^{\alpha t} - \alpha_0 \equiv \alpha_1 t + \alpha_2 t^2/2! + \dots$ , and we have denoted derivative notation  $y_{kt} = \partial_t^k y$ . For the spacial case when  $\alpha_r = r!x$ ,  $\alpha_k = 0$ ,  $k \neq r$ , then we have

$$Y_n = \xi_n(x, t, r).$$

The polynomials (3.3) are polynomials about the derivatives of function  $y$ , for example, the first three are

$$Y_0 = 1, \quad Y_1 = y_t, \quad Y_2 = y_t^2 + y_{2t}, \quad Y_3 = y_t^3 + 3y_t y_{2t} + y_{3t},$$

More recently Lambert et al generalized the third Bell polynomials as

$$Y_n = e^{-y} \partial_{x_1}^{n_1} \dots \partial_{x_m}^{n_m} e^y, \quad (3.3)$$

where  $y = y(x_1, \dots, x_m) : \mathbb{R}^m \rightarrow \mathbb{R}$  [31]-[33].



We now propose the following multi-dimensional and super extension to the ordinary Bell polynomials (3.1)-(3.3).

**Definition 1.** Let  $f = f(\mathbf{x}, \boldsymbol{\theta}) : \mathbb{R}_\Lambda^{m,n} \rightarrow \Lambda_0$  be a superdifferential bosonic function, the generalized super Bell polynomials (super  $Y$ -polynomials) is defined as follows

$$Y_{\boldsymbol{\ell} \cdot \mathbf{x}, \boldsymbol{\theta}}(f) = Y_{\boldsymbol{\ell} \cdot \mathbf{x}, \boldsymbol{\theta}}[f_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}] \equiv e^{-f} \mathcal{D}_1 \cdots \mathcal{D}_n \partial_{x_1}^{\ell_1} \cdots \partial_{x_m}^{\ell_m} e^f, \quad (3.4)$$

where  $\ell_j \geq 0$ ,  $j = 1, \dots, m$  denote arbitrary integers. To make subscript in expressions simple, we use some abbreviation notations in our context, for example,

$$\boldsymbol{\ell} = (\ell_1, \dots, \ell_m), \quad \boldsymbol{\theta} = (\theta_1, \dots, \theta_n), \quad \boldsymbol{\ell} \cdot \mathbf{x} = (\ell_1 x_1, \dots, \ell_m x_m),$$

$$\mathbf{r} = (r_1, \dots, r_m), \quad \mathbf{r} \cdot \mathbf{x} = (r_1 x_1, \dots, r_m x_m),$$

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_n), \quad \boldsymbol{\mu} \cdot \boldsymbol{\theta} = (\mu_1 \theta_1, \dots, \mu_n \theta_n).$$

**Remark 1.** The first notation  $Y_{\boldsymbol{\ell} \cdot \mathbf{x}, \boldsymbol{\theta}}(f)$  in (3.4) denotes the  $\ell_j$ -order derivatives of  $f$  with respect to the variable  $x_j$ ,  $j = 1, \dots, m$  and covariant derivatives with respect to  $\theta_k$ ,  $k = 1, \dots, n$ . The second notation  $Y_{\boldsymbol{\ell} \cdot \mathbf{x}, \boldsymbol{\theta}}[f_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}]$  implies that the super Bell polynomials (3.4) should be understood as such a multivariable differential polynomial with respect to partial derivatives  $f_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}$  ( $r_j = 0, \dots, \ell_j$ ,  $j = 1, \dots, m$ ,  $\mu_k = 0, 1$ ,  $k = 1, \dots, n$ ), but not variable elements  $x_j, \theta_k$  ( $j = 1, \dots, m$ ;  $k = 1, \dots, n$ ) as ordinary Bell polynomials. For instance, the  $Y_{3x}(f)$  in the next example is a polynomial  $Y_{3x}(f_x, f_{2x}, f_{3x})$  with respect to three variable elements  $f_x, f_{2x}, f_{3x}$ .

To better understanding our generalized super Bell polynomials, let us see an illustrative example. For the special case  $f = f(x, \theta_1, \theta_2)$ , the associated super

Bell polynomials defined by (3.4) read

$$Y_x(f) = f_x, \quad Y_{2x}(f) = f_{2x} + f_x^2,$$

$$Y_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3, \quad Y_{\theta_1}(f) = \mathcal{D}_1 f,$$

$$Y_{\theta_1 \theta_2}(f) = \mathcal{D}_1 \mathcal{D}_2 f + (\mathcal{D}_1 f) \mathcal{D}_2 f, \quad Y_{x, \theta_1}(f) = \mathcal{D}_1 f_x + f_x \mathcal{D}_1 f,$$

$$Y_{2x, \theta_1}(f) = f_{2x} \mathcal{D}_1 f + \mathcal{D}_1 f_{2x} + f_x^2 \mathcal{D}_1 f + 2f_x \mathcal{D}_1 f_x,$$

$$Y_{3x, \theta_1}(f) = f_{3x} \mathcal{D}_1 f + 3f_x f_{2x} \mathcal{D}_1 f + 3f_{2x} \mathcal{D}_1 f_x + 3f_x \mathcal{D}_1 f_{2x} + 3f_x^2 \mathcal{D}_1 f_x + \mathcal{D}_1 f_{3x}.$$

Let see the relations between our generalized super polynomials and ordinary Bell polynomials, as well as ordinary generalized Bell polynomials.

For the special case  $\mathbb{R}_\Lambda^{m,n} = \mathbb{R}^2$ ,  $\ell_2 = 0$ ,  $f = f(x_1, x_2) = x_2 x_1^r$  with the constant integer  $r > 0$ , then (3.4) reduces to the first Bell polynomials (3.1)

$$Y_{\ell_1 x_1}(f) = e^{-x_2 x_1^r} \partial_{x_1}^{\ell_1} e^{x_2 x_1^r} = \xi_{\ell_1}(x_1, x_2).$$

For the case  $\mathbb{R}_\Lambda^{m,n} = \mathbb{R}^m$ , the corresponding generalized super Bell polynomials (3.4) degenerates to generalized Bell polynomials (3.3) given by Lambert et al. The Bell polynomials admit partitional representation [31]

$$Y_{\ell \cdot x}(f) = \sum \frac{\ell_1! \cdots \ell_m!}{c_1! \cdots c_k!} \prod_{j=1}^k \left( \frac{f_{r_{1j}, \dots, r_{mj}}}{r_{1j}! \cdots r_{mj}!} \right)^{c_j}, \quad (3.5)$$

where the sum is to taken over all partitions  $[(r_{j1}, \dots, r_{m1})^{c_1}, \dots, (r_{jk}, \dots, r_{mk})^{c_k}]$  the  $m$ -tuple  $(\ell_1, \dots, \ell_m)$ .

In following, we investigate properties of super Bell polynomials which are key results to establish connects with supersymmetric equations.

**Theorem 1.** Under the Hopf-Cole transformation  $f = \ln \psi$ , the generalized super Bell polynomials  $Y_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}(f)$  can be “linearized” into the form

$$Y_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}(f)|_{f=\ln \psi} = \psi_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}} / \psi. \quad (3.6)$$

*Proof.* According to the definition (3.4), we have

$$Y_{\mathbf{r}, \mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\theta}}(f)|_{f=\ln \psi} = e^{-\ln \psi} \mathcal{D}_1^{\mu_1} \cdots \mathcal{D}_n^{\mu_n} \partial_{x_1}^{r_1} \cdots \partial_{x_m}^{r_m} e^{\ln \psi} = \psi_{\mathbf{r}, \mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\theta}} / \psi,$$

which finishes the proof of Theorem 1.  $\square$

**Remark 2.** According to the theorem, under the Hopf-Cole transformation  $f = \ln \psi$ , a equation in term of linear combination of super Bell polynomials, i.e.

$$\sum_{\mathbf{r}, \boldsymbol{\mu}} C_{\mathbf{r}, \boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\theta}) Y_{\mathbf{r}, \mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\theta}}(f) = 0$$

can be linearized into the form

$$\sum_{\mathbf{r}, \boldsymbol{\mu}} C_{\mathbf{r}, \boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\theta}) \psi_{\mathbf{r}, \mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\theta}} = 0,$$

where  $C_{\mathbf{r}, \boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\theta})$  are functions independent of the function  $f$ . This is a key property to construct the Lax pair of supersymmetric equations.

**Theorem 2.** The super Bell polynomials (3.4) admit recursion formula

$$Y_{\ell, \mathbf{x}, \boldsymbol{\theta}}(f) = \prod_{k=1}^n (\mathcal{D}_k + \mathcal{D}_k f) Y_{\ell, \mathbf{x}}(f). \quad (3.7)$$

*Proof.* By the definition (3.4), direct computation leads to

$$\begin{aligned} Y_{\ell, \mathbf{x}, \boldsymbol{\theta}}(f) &= \mathcal{D}_1 Y_{\ell, \mathbf{x}, \theta_2, \dots, \theta_n}(f) + (\mathcal{D}_1 f) Y_{\ell, \mathbf{x}, \theta_2, \dots, \theta_n}(f) \\ &= (\mathcal{D}_1 + \mathcal{D}_1 f) Y_{\ell, \mathbf{x}, \theta_2, \dots, \theta_n}(f). \end{aligned}$$

Similarly,

$$Y_{\ell, \mathbf{x}, \theta_2, \dots, \theta_n}(f) = (\mathcal{D}_2 + \mathcal{D}_2 f) Y_{\ell, \mathbf{x}, \theta_3, \dots, \theta_n}(f).$$

Repeating the above arguments then proves the formula (3.7).  $\square$

**Theorem 3.** The super Bell polynomials (3.4) possess parity property

$$Y_{\ell, \mathbf{x}, \boldsymbol{\theta}}[(-1)^{\sum r_j + \sum \mu_k} f_{\mathbf{r}, \mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\theta}}] = (-1)^{n + \sum \ell_j} Y_{\ell, \mathbf{x}, \boldsymbol{\theta}}[f_{\mathbf{r}, \mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\theta}}]. \quad (3.8)$$

*Proof.* From the recursion relation (3.7), it follows that

$$\begin{aligned} Y_{\ell, \mathbf{x}, \boldsymbol{\theta}}[(-1)^{\sum r_j + \sum \mu_k} f_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}] &= \prod_{k=1}^n (-\mathcal{D}_k - \mathcal{D}_k f) Y_{\ell, \mathbf{x}}[(-1)^{\sum r_j} f_{\mathbf{r} \cdot \mathbf{x}}] \\ &= (-1)^n \prod_{k=1}^n (\mathcal{D}_k + \mathcal{D}_k f) Y_{\ell, \mathbf{x}}[(-1)^{\sum r_j} f_{\mathbf{r} \cdot \mathbf{x}}]. \end{aligned} \quad (3.9)$$

While applying the partitional representation (3.5), we have

$$Y_{\ell, \mathbf{x}}[(-1)^{\sum r_j} f_{\mathbf{r} \cdot \mathbf{x}}] = (-1)^{\sum \ell_j} Y_{\ell, \mathbf{x}}[f_{\mathbf{r} \cdot \mathbf{x}}]. \quad (3.10)$$

Hence, combining (3.7), (3.9) and (3.10) proves the formula (3.8).  $\square$

**Theorem 4.** The super Bell polynomials (3.4) obey addition property

$$\begin{aligned} Y_{\ell, \mathbf{x}, \boldsymbol{\theta}}(f + g) &= \sum_{\mu_1, \dots, \mu_n=0}^1 (-1)^{\tau[\{(1-\boldsymbol{\mu}) \cdot \mathbf{n}, \boldsymbol{\mu} \cdot \mathbf{n}\} \setminus \{0\}]} \sum_{r_1=0}^{\ell_1} \dots \sum_{r_m=0}^{\ell_m} \prod_{i=1}^m \binom{\ell_i}{r_i} \\ &\times Y_{(\ell-\mathbf{r}) \cdot \mathbf{x}, (1-\boldsymbol{\mu}) \cdot \boldsymbol{\theta}}(f) Y_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}(g), \end{aligned} \quad (3.11)$$

where  $\mathbf{n} = (1, 2, \dots, n)$ ;  $\tau[\{(1-\boldsymbol{\mu}) \cdot \mathbf{n}, \boldsymbol{\mu} \cdot \mathbf{n}\} \setminus \{0\}]$  denotes the reverse order numbers of the  $n$ -order permutation  $\{(1-\boldsymbol{\mu}) \cdot \mathbf{n}, \boldsymbol{\mu} \cdot \mathbf{n}\} \setminus \{0\}$ , which is generated from anticommutation of covariant derivatives  $\mathcal{D}_j$ ,  $j = 1, \dots, n$ , and obtained from a  $2n$ -order permutation  $\{(1-\mu_1)1, \dots, (1-\mu_n)n, \mu_1 1, \dots, \mu_n n\}$  ( $\mu_j = 0$  or 1) by taking off all zero terms. The  $2n$ -order permutation is the subscript of corresponding covariant derivatives of term  $Y_{(\ell-\mathbf{r}) \cdot \mathbf{x}, (1-\boldsymbol{\mu}) \cdot \boldsymbol{\theta}}(f) Y_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}(g)$  kept in original order.

*Proof.* According to the commutative properties of covariant derivatives (3.1), a minus sign in the Leibnitz rule exactly corresponds to an inverse order of the  $n$ -order permutation  $\{1, 2, \dots, n\}$ . So direct computation shows that

$$\begin{aligned} (FG)^{-1} \mathcal{D}_1 \dots \mathcal{D}_n \partial_{x_1}^{\ell_1} \dots \partial_{x_m}^{\ell_m} (FG) &= \sum_{\mu_1, \dots, \mu_n=0}^1 (-1)^{\tau[\{(1-\boldsymbol{\mu}) \cdot \mathbf{n}, \boldsymbol{\mu} \cdot \mathbf{n}\} \setminus \{0\}]} \sum_{r_1=0}^{\ell_1} \dots \sum_{r_m=0}^{\ell_m} \prod_{i=1}^m \binom{\ell_i}{r_i} \\ &\times (F \mathcal{D}_1^{1-\mu_1} \dots \mathcal{D}_n^{1-\mu_n} \partial_{x_1}^{\ell_1-r_1} \dots \partial_{x_m}^{\ell_m-r_m} F) (G \mathcal{D}_1^{\mu_1} \dots \mathcal{D}_n^{\mu_n} \partial_{x_1}^{r_1} \dots \partial_{x_m}^{r_m} G), \end{aligned}$$

which implies (3.11) by replacing  $F = e^f$  and  $G = e^g$ .  $\square$

Let  $F, G : \mathbb{R}_\Lambda^{m,n} \rightarrow \Lambda_0$  be two bosonic superdifferential functions, then direct computation yields

$$\begin{aligned} \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 (FG) &= (\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 F)G + (\mathcal{D}_2 \mathcal{D}_3 F)\mathcal{D}_1 G - (\mathcal{D}_1 \mathcal{D}_3 F)\mathcal{D}_2 G \\ &+ (\mathcal{D}_2 F)(\mathcal{D}_1 F)\mathcal{D}_2 G + (\mathcal{D}_1 \mathcal{D}_2 F)\mathcal{D}_3 G - (\mathcal{D}_2 F)(\mathcal{D}_1 F)\mathcal{D}_3 G \\ &+ (\mathcal{D}_1 F)(\mathcal{D}_2 F)\mathcal{D}_3 G + F(\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 G), \end{aligned}$$

in which corresponding six even permutations are  $\{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}$ , and two odd permutations are  $\{1, 3, 2\}, \{2, 1, 3\}$ .

## 2.2. Generalized super binary Bell polynomials

We further define a class of super binary Bell polynomials which play an important role in the study of integrability for supersymmetric equations.

**Definition 2.** Based on the use of above super Bell polynomials (3.4), the super binary Bell polynomials (  $\mathcal{Y}$ -polynomials) can be defined as follows

$$\mathcal{Y}_{\ell \cdot \mathbf{x}, \boldsymbol{\theta}}(v, w) = Y_{\ell \cdot \mathbf{x}, \boldsymbol{\theta}}[f_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}], \quad (3.12)$$

in which we replace the function  $f$  and its derivatives by corresponding terms of functions  $w$  and  $v$  respectively, according the following rule

$$f_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}} = \begin{cases} v_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}, & \text{if } \sum_{j=1}^m r_j + \sum_{k=1}^n \mu_k \text{ is odd,} \\ w_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}, & \text{if } \sum_{j=1}^m r_j + \sum_{k=1}^n \mu_k \text{ is even,} \end{cases}$$

The super binary Bell polynomials (3.6) is multi-variable polynomials with respect to various partial derivatives  $v_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}$  and  $w_{\mathbf{r} \cdot \mathbf{x}, \boldsymbol{\mu} \cdot \boldsymbol{\theta}}$ ,  $r_j = 0, \dots, \ell_j$ ,  $j = 0, \dots, m, \mu_k = 0, 1, k = 1, \dots, n$ .

The super binary Bell polynomials also inherit the easily recognizable partial structure of the super Bell polynomials. The first few are explicitly calculated as

$$\begin{aligned}
\mathcal{Y}_x(v) &= v_x, \quad \mathcal{Y}_{2x}(v, w) = w_{2x} + v_x^2, \quad \mathcal{Y}_{3x}(v, w) = v_{3x} + 3v_x w_{2x} + v_x^3, \\
\mathcal{Y}_{\theta_1}(v) &= \mathcal{D}_1 v, \quad \mathcal{Y}_{\theta_1 \theta_2}(w, v) = \mathcal{D}_1 \mathcal{D}_2 w + (\mathcal{D}_1 v) \mathcal{D}_2 v, \\
\mathcal{Y}_{x, \theta_1}(v, w) &= \mathcal{D}_1 w_x + v_x \mathcal{D}_1 v, \\
\mathcal{Y}_{2x, \theta_1}(v, w) &= w_{2x} \mathcal{D}_1 v + \mathcal{D}_1 v_{2x} + v_x^2 \mathcal{D}_1 v + 2v_x \mathcal{D}_1 w_x, \\
\mathcal{Y}_{3x, \theta_1}(v, w) &= v_{3x} \mathcal{D}_1 v + 3v_x w_{2x} \mathcal{D}_1 v + 3w_{2x} \mathcal{D}_1 w_x + 3v_x \mathcal{D}_1 v_{2x} \\
&\quad + 3v_x^2 \mathcal{D}_1 w_x + \mathcal{D}_1 w_{3x}.
\end{aligned} \tag{3.13}$$

We denote the special case of super Bell polynomials by  $\mathcal{Y}_{\ell, \mathbf{x}, \boldsymbol{\theta}}(v = 0, w) = P_{\ell, \mathbf{x}, \boldsymbol{\theta}}(w)$ , then it follows from (3.13) that

$$\begin{aligned}
P_{2x}(w) &= w_{2x}, \quad P_{4x}(w) = w_{4x} + 3w_{2x}^2, \quad P_{\theta_1, \theta_2}(w) = \mathcal{D}_1 \mathcal{D}_2 w, \\
P_{x, \theta_1}(w) &= \mathcal{D}_1 w_x, \quad P_{3x, \theta_1}(w) = \mathcal{D}_1 w_{3x} + 3w_{2x} \mathcal{D}_1 w_x, \dots
\end{aligned} \tag{3.14}$$

**Theorem 5.** The link between super binary Bell polynomials  $\mathcal{Y}_{\ell, \mathbf{x}, \boldsymbol{\theta}}(v, w)$  and the super Hirota bilinear equation  $S_1 \cdots S_n D_{x_1}^{\ell_1} \cdots D_{x_m}^{\ell_m} F \cdot G$  can be established by an identity

$$\mathcal{Y}_{\ell, \mathbf{x}, \boldsymbol{\theta}}(v = \ln F/G, w = \ln FG) = (FG)^{-1} S_1 \cdots S_n D_{x_1}^{\ell_1} \cdots D_{x_m}^{\ell_m} F \cdot G. \tag{3.15}$$

This formula will be used to obtain bilinear Bäcklund transformations of super-symmetric equations.

*Proof.* Let  $f = \ln F$ ,  $g = \ln G$ , then we have  $v = f - g$ ,  $w = f + g$ . It follows from the definition 2 that

$$\begin{aligned}
\mathcal{Y}_{\ell, \mathbf{x}, \boldsymbol{\theta}}(v = \ln F/G, w = \ln FG) &= Y_{\ell, \mathbf{x}, \boldsymbol{\theta}}[f_{\mathbf{s} \cdot \mathbf{x}, \tilde{\boldsymbol{\mu}} \boldsymbol{\theta}} + (-1)^{\sum s_i + \sum \tilde{\mu}_j} g_{\mathbf{s} \cdot \mathbf{x}, \tilde{\boldsymbol{\mu}} \boldsymbol{\theta}}] \\
&\stackrel{(3.11)}{=} \sum_{\mu_1, \dots, \mu_n=0}^1 (-1)^{\tau[\{(1-\mu) \cdot \mathbf{n}, \mu \cdot \mathbf{n}\}/0]} \sum_{r_1=0}^{\ell_1} \dots \sum_{r_m=0}^{\ell_m} \prod_{i=1}^m \binom{\ell_i}{r_i} \\
&\quad \times Y_{(\ell-r) \cdot \mathbf{x}, (1-\mu) \cdot \boldsymbol{\theta}}[f_{\mathbf{s} \cdot \mathbf{x}, \tilde{\boldsymbol{\mu}} \boldsymbol{\theta}}] Y_{\mathbf{r} \cdot \mathbf{x}, \mu \cdot \boldsymbol{\theta}}[(-1)^{\sum s_i + \sum \tilde{\mu}_j} g_{\mathbf{s} \cdot \mathbf{x}, \tilde{\boldsymbol{\mu}} \boldsymbol{\theta}}] \\
&\stackrel{(3.8)}{=} \sum_{\mu_1, \dots, \mu_n=0}^1 (-1)^{\tau[\{(1-\mu) \cdot \mathbf{n}, \mu \cdot \mathbf{n}\}/0] + \sum_{j=1}^m r_j + \sum_{k=1}^n \mu_k} \sum_{r_1=0}^{\ell_1} \dots \sum_{r_m=0}^{\ell_m} \prod_{i=1}^m \binom{\ell_i}{r_i} \\
&\quad \times Y_{(\ell-r) \cdot \mathbf{x}, (1-\mu) \cdot \boldsymbol{\theta}}(f) Y_{\mathbf{r} \cdot \mathbf{x}, \mu \cdot \boldsymbol{\theta}}(g) \\
&= (FG)^{-1} S_1 \dots S_n D_{x_1}^{\ell_1} \dots D_{x_m}^{\ell_m} F \cdot G.
\end{aligned}$$

□

For the particular case when  $F = G$ , the formula (3.12) reduces to

$$\begin{aligned}
G^{-2} S_1 \dots S_n D_{x_1}^{\ell_1} \dots D_{x_m}^{\ell_m} G \cdot G &= \mathcal{Y}_{\ell, \mathbf{x}, \boldsymbol{\theta}}(0, w = 2 \ln G) \\
&= \begin{cases} 0, & n + \sum_{j=1}^m \ell_j \text{ is odd,} \\ P_{\ell, \mathbf{x}, \boldsymbol{\theta}}(w), & n + \sum_{j=1}^m \ell_j \text{ is even,} \end{cases} \quad (3.16)
\end{aligned}$$

which implies that the  $P$ -polynomials can be characterized by an equally recognizable even part partitional structure. The formulae (3.15) and (3.16) will prove particularly useful in connecting supersymmetric equations with their corresponding super bilinear equations. Once a nonlinear equation is expressible as a linear combination of super Bell  $\mathcal{Y}$ -polynomials or  $P$ -polynomials, then it can be transformed into a super linear equation.

**Theorem 6.** The super binary Bell polynomials  $\mathcal{Y}_{\ell \cdot x, \theta}(v, w)$  can be separated into super  $P$ -polynomials and super Bell  $Y$ -polynomials

$$\begin{aligned} \mathcal{Y}_{\ell \cdot x, \theta}(v, w) = & \sum_{\mu_1, \dots, \mu_n=0}^1 (-1)^{\tau[\{(1-\mu) \cdot n, \mu \cdot n\}/0]} \sum_{r_1=0}^{\ell_1} \dots \sum_{r_m=0}^{\ell_m} \prod_{i=1}^m \binom{\ell_i}{r_i} \\ & \times P_{r \cdot x, \mu \cdot \theta}(w - v) Y_{(\ell-r) \cdot x, (1-\mu) \cdot \theta}(v), \end{aligned} \quad (3.17)$$

where only non-vanishing contributions being those for which  $\sum r_j + \sum \mu_k$  is even integer.

*Proof.* According Definition 2 of the super Bell polynomials, we have

$$\mathcal{Y}_{p \cdot x, \nu \cdot \theta}(v = 0, w) = 0, \quad \text{as } \sum p_j + \sum \nu_k \text{ is odd,}$$

so that by using Theorem 4,

$$\begin{aligned} \mathcal{Y}_{\ell \cdot x, \theta}(v, w) &= \mathcal{Y}_{\ell \cdot x, \theta}(v, v + q) = Y_{\ell \cdot x, \theta}(v + q)|_{q_{p \cdot x, \nu \cdot \theta}=0} \\ &= \sum_{\mu_1, \dots, \mu_n=0}^1 (-1)^{\tau[\{(1-\mu) \cdot n, \mu \cdot n\}/0]} \sum_{r_1=0}^{\ell_1} \dots \sum_{r_m=0}^{\ell_m} \prod_{i=1}^m \binom{\ell_i}{r_i} \\ &\times Y_{r \cdot x, \mu \cdot \theta}(q) Y_{(\ell-r) \cdot x, (1-\mu) \cdot \theta}(v)|_{q_{p \cdot x, \nu \cdot \theta}=0}, \end{aligned} \quad (3.18)$$

where  $q = w - v$ , the sum  $\sum p_j + \sum \nu_k$  is odd integer.

Substituting the relation

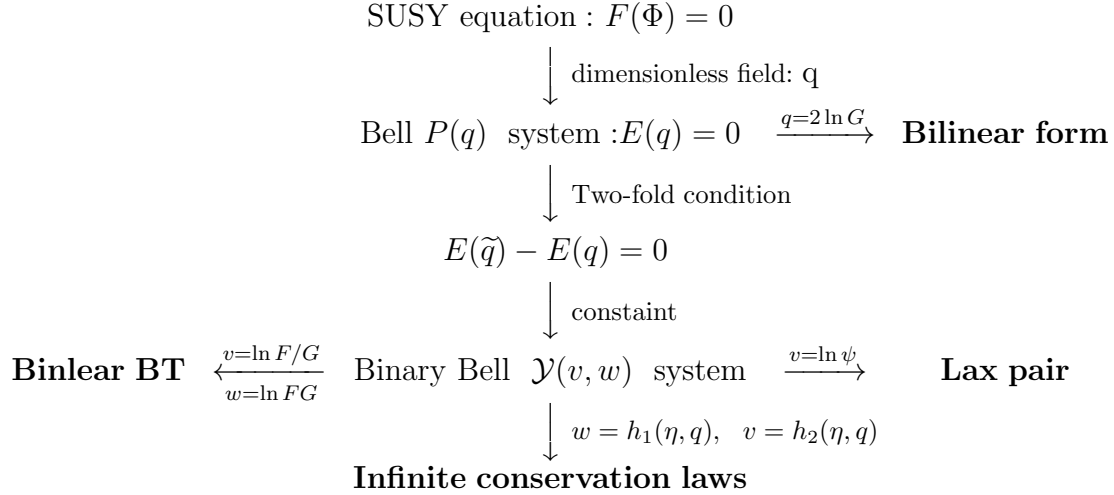
$$Y_{r \cdot x, \mu \cdot \theta}(q)|_{q_{p \cdot x, \nu \cdot \theta}=0, \sum p_j + \sum \nu_k \text{ is odd}} = P_{r \cdot x, \mu \cdot \theta}(q)|_{\sum p_j + \sum \nu_k \text{ is even}},$$

into (3.18) then leads to the formula (3.17).  $\square$

This theorem implies that the super binary Bell polynomials (3.12) can still be “linearized” by means of the Hopf-Cole transformation  $v = \ln \psi$ ,  $\psi = F/G$ . The formulae (3.6) and (3.17) will then provide a way to find the associated Lax system of supersymmetric equations.

Finally, let's through a graph describe general procedure how to use the theory of super Bell polynomials that we have developed above.





It is clear from this graph to see the close connections among Bell polynomials with bilinear equation, bilinear Bäcklund transformation, Lax pair and conservation laws.

#### 4. The supersymmetric KdV equation

Consider the supersymmetric KdV equation of Manin-Radul-Mathieu [4, 6]

$$\Phi_t + 3(\Phi \mathcal{D}\Phi)_x + \Phi_{3x} = 0, \quad (4.1)$$

where  $\Phi = \Phi(x, t, \theta) : \mathbb{R}_\Lambda^{2,1} \rightarrow \Lambda_1$  is a fermionic super field function with independent variables  $x$ ,  $t$  and Grassmann variable  $\theta$ . The symbol  $\mathcal{D} = \partial_\theta + \theta \partial_x$  denotes the super derivative differential operator, which satisfies  $\mathcal{D}^2 = \partial_x$ ,  $\theta^2 = 0$ . The supersymmetric version of the KdV equation (4.1) describe the time evolution of a Grassmann-valued superfield  $\Phi(x, t, \theta) = \tilde{u}(x, t) + \theta u(x, t)$ , where  $u(x, t)$  is an ordinary function and  $\tilde{u}(x, t)$  is a Grassmann valued function. The variable  $x, t$  acquire a Grassmann partner  $\theta$ , so  $(x, t, \theta)$  are coordinates in a one dimensional

superspace  $\mathbb{R}_\Lambda^{2,1}$ . Since the introduction of the supersymmetric KdV equation (1.1) by Manin, Radul and Mathieu [4, 6], much attention has been given to its mathematical structure and integrable properties. For instances, bi-Hamiltonian structure, Painlevé property, infinite many symmetries, Darboux transformation, Bäcklund transformation, bilinear form, super soliton solutions and super quasi-periodic solutions had been investigated in [10]–[21]. Here we see how to apply the super polynomials to investigate complete integrability of the supersymmetric KdV equation (4.1).

**Theorem 7.** Under the transformation  $\Phi = 2\mathcal{D}(\ln G)_x$ , the supersymmetric KdV equation (4.1) can be bilinearized into

$$(SD_t + SD_x^3)G \cdot G = 0. \quad (4.2)$$

*Proof.* The invariance of the equation (4.1) under the scale transformation

$$x \rightarrow \lambda x, \quad t \rightarrow \lambda^3 t, \quad \theta \rightarrow \lambda^{1/2} \theta, \quad \Phi \rightarrow \lambda^{-3/2} \Phi$$

shows that the dimension of the fermionic field  $\Phi$  is  $-3/2$ , and it can be related to a dimensionless bosonic field  $q : \mathbb{R}_\Lambda^{2,1} \rightarrow \Lambda_0$ , by setting

$$\Phi = c\mathcal{D}q_x, \quad (4.3)$$

with  $c \in \Lambda_0$  being free function to be the appropriate choice such that the equation (4.1) connects with  $P$ -polynomials. Substituting (4.3) into (4.1) and integrating with respect to  $x$  yields

$$E(q) \equiv \mathcal{D}q_t + \mathcal{D}q_{3x} + 3cq_{2x}\mathcal{D}_\theta q_x = 0. \quad (4.4)$$

Comparing the last two terms of this equation with the formula (3.16) implies that we should require  $c = 1$ . The equation (4.4) is then cast into a combination

form of  $P$ -polynomials

$$E(q) = P_{t,\theta}(q) + P_{3x,\theta}(q) = 0. \quad (4.5)$$

Making a change of dependent variable

$$q = 2 \ln G, \quad \Longleftrightarrow \quad \Phi = 2\mathcal{D}(\ln G)_x$$

with  $G : \mathbb{R}_\Lambda^{2,1} \rightarrow \Lambda_0$ , then the property (3.16) shows that the equation (4.5) is equivalent to the bilinear equation (4.2).  $\square$

Starting from the bilinear equation (4.2), it is easy to get super soliton solutions. For example, the regular one-soliton like solution reads

$$\Phi = \mathcal{D}[\ln(1 + \exp(kx - k^3t + \theta\zeta))],$$

where  $k \in \Lambda_0, \zeta \in \Lambda_1$ . Since solving the equation (4.1) is not our main purpose in this paper, the super soliton solutions can be found in details [20].

Next, we search for the bilinear Bäcklund transformation and Lax pair of the supersymmetric KdV equation (4.1).

**Theorem 8.** Let  $F$  be a solution of the equation (4.2), then  $G$  satisfying

$$\begin{aligned} (SD_x - \lambda S)F \cdot G &= 0, \\ (D_t + D_x^3 + 3\lambda^2 D_x - 3\lambda D_x)F \cdot G &= 0 \end{aligned} \quad (4.5)$$

is another solution of the equation (4.2). This kind of Bäcklund transformation is exactly the same with that given by Liu [17]

*Proof.* Let  $q = 2 \ln G$ ,  $\tilde{q} = 2 \ln F : \mathbb{R}_\Lambda^{2,1} \rightarrow \Lambda_0$  be two different solutions of the equation (4.4), respectively, we associate the two-field condition

$$E(\tilde{q}) - E(q) = \mathcal{D}(\tilde{q} - q)_t + \mathcal{D}(\tilde{q} - q)_{3x} + 3\tilde{q}_{2x}\mathcal{D}\tilde{q}_x - 3q_{2x}\mathcal{D}q_x = 0. \quad (4.6)$$

This two-field condition can be regarded as a ansatz for a bilinear Bäcklund transformation and may produce the required transformation under appropriate additional constraints.

To find such constraints, we introduce two new variables

$$v = (\tilde{q} - q)/2 = \ln F/G, \quad w = (\tilde{q} + q)/2 = \ln FG, \quad (4.7)$$

and rewrite the condition (4.6) into the form

$$\begin{aligned} E(\tilde{q}) - E(q) &= 2\mathcal{D}v_t + 2\mathcal{D}v_{3x} + 6v_{2x}\mathcal{D}w_x + 6w_{2x}\mathcal{D}v_x \\ &= 2\mathcal{D}[\mathcal{Y}_t(v) + \mathcal{Y}_{3x}(v, w)] + 6R(v, w) = 0, \end{aligned} \quad (4.8)$$

with

$$R(v, w) = v_{2x}\mathcal{D}w_x - v_x\mathcal{D}w_{2x} - v_x^2\mathcal{D}v_x = \text{Wronskian}[\mathcal{Y}_{x,\theta}(v, w), \mathcal{Y}_x(v)].$$

In order to decouple the two-field condition (4.8) into a pair of constraints, we impose such a constraint which enable us to express  $R(v, w)$  as the  $\mathcal{D}$ -derivative of a combination of  $\mathcal{Y}$ -polynomials. A possible choice of such constraint may be

$$\mathcal{Y}_{x,\theta}(v, w) = \lambda\mathcal{Y}_\theta(v), \quad (4.9)$$

where  $\lambda \in \Lambda_0$  is an arbitrary parameter. It follows from the identity (4.9) that

$$(v_x\mathcal{D}v)_x = \lambda\mathcal{D}_\theta v_x - \mathcal{D}w_{2x},$$

on account which, then  $R(v, w)$  can be rewritten in the form

$$\begin{aligned} R(v, w) &= \lambda(v_x\mathcal{D}v)_x - 2\lambda v_x\mathcal{D}v_x = \lambda^2\mathcal{D}v_x - \lambda\mathcal{D}w_{2x} - 2\lambda v_x\mathcal{D}v_x \\ &= \mathcal{D}[\lambda^2\mathcal{Y}_x(v) - \lambda\mathcal{Y}_{2x}(v, w)]. \end{aligned} \quad (4.10)$$

Then from (4.7)-(4.10), we deduce a coupled system of super binary Bell  $\mathcal{Y}$ -polynomials

$$\begin{aligned} \mathcal{Y}_{x,\theta}(v, w) - \lambda\mathcal{Y}_\theta(v) &= 0, \\ \mathcal{Y}_t(v) + \mathcal{Y}_{3x}(v, w) + 3\lambda^2\mathcal{Y}_x(v) - 3\lambda\mathcal{Y}_{2x}(v, w) &= 0. \end{aligned} \quad (4.11)$$

By application of the identity (3.15), under transformation  $v = \ln F/G, w = \ln FG$ , the system (4.11) then leads to the bilinear Bäcklund transformation (4.5).

□

**Theorem 9.** The supersymmetric KdV equation (4.1) admits a Lax pair

$$\begin{aligned} (\partial_x^2 + \Phi \mathcal{D} - \lambda \partial_x) \varphi &= 0, \\ [\mathcal{D} \partial_t + \mathcal{D} \partial_x^3 - 3\lambda \mathcal{D} \partial_x^2 + 3(\mathcal{D} \Phi + \lambda) \mathcal{D} + (\mathcal{D} \Phi) \mathcal{D}] \varphi &= 0, \end{aligned} \quad (4.12)$$

where  $\varphi : \mathbb{R}_\Lambda^{2,1} \rightarrow \Lambda_1$  is a fermionic eigenfunction.

*Proof.* By transformation  $v = \ln \psi$ , it follows from the formulae (3.6) and (3.17) that

$$\mathcal{Y}_\theta(v) = \mathcal{D}\psi/\psi, \quad \mathcal{Y}_{x,\theta}(v, w) = \mathcal{D}q_x + \mathcal{D}\psi_x/\psi,$$

$$\mathcal{Y}_t(v) = \psi_t/\psi, \quad \mathcal{Y}_{2x}(v, w) = q_{2x} + \psi_{2x}/\psi, \quad \mathcal{Y}_{3x}(v, w) = 3q_{2x}\psi_x/\psi + \psi_{3x}/\psi,$$

on account of which, the system (4.12) is then linearized into a Lax pair with a parameter  $\lambda$

$$L_1 \psi \equiv (\mathcal{D} \partial_x - \lambda \mathcal{D} + \mathcal{D} q_x) \psi = 0,$$

$$L_2 \psi \equiv (\partial_t + \partial_x^3 + 3q_{2x} \partial_x + 3\lambda^2 \partial_x - 3\lambda \partial^2 + q_{2x}) \psi = 0,$$

which is equivalent to the formula (4.12) b by replacing  $\mathcal{D}q_x$  with  $\Phi$ , and  $\psi$  with  $\mathcal{D}\varphi$ . It is easy to check that the integrability condition of the Lax pair

$$[L_1, L_2] \psi = 0$$

is satisfied if  $\Phi$  is a solution of the supersymmetric KdV equation (4.1). □

Finally, we show how to derive the infinite conservation laws for super KdV equation (4.1) based on the use of the binary Bell polynomials.

**Theorem 10.** The supersymmetric KdV equation (4.1) possesses the following infinite conservation laws

$$I_{n,t} + \mathcal{D}F_n = 0, \quad n = 1, 2, \dots \quad (4.13)$$

where the fermionic conserved densities  $I'_n$ s are explicitly given by recursion relations

$$\begin{aligned} I_1 &= \mathcal{D}q_x = \Phi, \quad I_2 = I_{1,x} = \Phi_x, \\ I_{n+1} &= I_{n,x} + \sum_{k=1}^n I_k \mathcal{D}I_{n-k}, \quad n = 2, 3, \dots, \end{aligned} \quad (4.14)$$

and the bosonic fluxes  $F'_n$ s are given by recursion formulas

$$\begin{aligned} F_1 &= \mathcal{D}\Phi_{2x} + 3\Phi\Phi_x + 3(\mathcal{D}\Phi)^2, \\ F_2 &= \mathcal{D}\Phi_{3x} + 3(\Phi\Phi_{2x} + \Phi_x^2) + 6\mathcal{D}\Phi\mathcal{D}\Phi_x, \\ F_n &= \mathcal{D}I_{n,2x} + 3 \sum_{k=1}^n (I_k I_{n+1-k} + \mathcal{D}I_k \mathcal{D}I_{n+1-k,x}) + 3\mathcal{D}\Phi\mathcal{D}I_n \\ &\quad + \sum_{i+j+k=n} \mathcal{D}I_i \mathcal{D}I_j \mathcal{D}I_k, \quad n = 3, 4, \dots. \end{aligned} \quad (4.15)$$

*Proof.* The conservation laws actually have been hinted in the two-filed constraint system (4.9)-(4.11), which can be rewritten in the conserved form

$$\mathcal{Y}_{x,\theta}(v, w) - \lambda \mathcal{Y}_\theta(v) = 0, \quad (4.16)$$

$$\partial_t \mathcal{Y}_\theta(v) + \mathcal{D}[\mathcal{Y}_{3x}(v, w) + 3\lambda^2 \mathcal{Y}_x(v) - 3\lambda \mathcal{Y}_{2x}(v, w)] = 0.$$

by applying the relation  $\mathcal{D}\mathcal{Y}_t(v) = \partial_t \mathcal{Y}_\theta(v) = \mathcal{D}v_t$ .

By introducing a new fermionic potential function

$$\eta = (\mathcal{D}\tilde{q} - \mathcal{D}q)/2 : \mathbb{R}_\Lambda^{2,1} \rightarrow \Lambda_1,$$

it follows from the relation (4.8) that

$$\mathcal{D}v = \eta, \quad \mathcal{D}w = \eta + \mathcal{D}q. \quad (4.17)$$

Substituting (4.17) into (4.16), we get a super Riccati-type equation

$$\eta_x + \eta \mathcal{D}\eta + \mathcal{D}q_x - \lambda \eta = 0, \quad (4.18)$$

and a divergence-type equation

$$\eta_t + \mathcal{D}[\mathcal{D}\eta_{2x} + 3\lambda \eta \eta_x + 3q_{2x} \mathcal{D}\eta + 3\mathcal{D}\eta \mathcal{D}\eta_x + (\mathcal{D}\eta)^3] = 0, \quad (4.19)$$

where we have used the equation (4.18) to get the equation (4.19).

To proceed, inserting the expansion

$$\eta = \sum_{n=1}^{\infty} I_n(\mathcal{D}q, q_x, \dots) \lambda^{-n}, \quad (4.20)$$

into the equation (4.18) and equating the coefficients for power of  $\lambda$ , we then obtain the formulas (4.13).

Finally, substituting (4.20) into (4.19) yields

$$\begin{aligned} & \sum_{n=1}^{\infty} I_{n,t} \lambda^{-n} + \mathcal{D} \left[ \sum_{n=1}^{\infty} \mathcal{D} I_{n,2x} \varepsilon^{-n} + 3\lambda \sum_{n=1}^{\infty} I_n \lambda^{-n} \sum_{n=1}^{\infty} I_{n,x} \lambda^{-n} + 3q_{2x} \sum_{n=1}^{\infty} \mathcal{D} I_n \lambda^{-n} \right. \\ & \left. + 3 \sum_{n=1}^{\infty} \mathcal{D} I_n \lambda^{-n} \sum_{n=1}^{\infty} \mathcal{D} I_{n,x} \lambda^{-n} + \left( \sum_{n=1}^{\infty} \mathcal{D} I_n \lambda^{-n} \right)^3 \right] = 0, \end{aligned}$$

which leads to infinite consequence of conservation law equation (4.13) by equating the coefficients for power of  $\lambda$ .  $\square$

It follows from the conservation equation (4.13) by using (2.2) that

$$\left( \iint I_n dx d\theta \right)_t = - \iint (\mathcal{D} F_n) dx d\theta = 0,$$

which implies that  $I'_n$ s are fermionic conserved densities. We present recursion formulas for generating an infinite sequence of conservation laws for each equation, the first few conserved density and associated flux are explicit. The first equation of conservation law equation (4.13) is exactly the supersymmetric KdV equation (4.1). In conclusion, the supersymmetric KdV (4.1) is complete integrable in the sense that it admits bilinear Bäcklund transformation, Lax pair and infinite conservation laws.

## 5. The supersymmetric sine-Gordon equation

The classical sine-Gordon equation

$$\phi_{xt} = \sin \phi \quad (5.1)$$

has applications in various areas of physics including nonlinear field theory, solid-state physics, nonlinear optics, elementary particle theory and fluid dynamics, see [38]-[42] and references therein. The supersymmetric extension of the equation (5.1), i.e. the supersymmetric sine-Gordon equation [43]-[50]

$$\mathcal{D}_1 \mathcal{D}_2 \Phi = \sin \Phi \quad (5.2)$$

is constructed on the four dimensional superspace  $(x, t, \theta_1, \theta_2) \in \mathbb{R}_\Lambda^{2,2}$ . Here,  $\Phi = \Phi(x, t, \theta_1, \theta_2) : \mathbb{R}_\Lambda^{2,2} \rightarrow \Lambda_0$  is a scalar bosonic superfield; The variables  $x$  and  $t$  represent the even coordinates on the two-dimensional super-Minkowski space, while the quantities  $\theta_1$  and  $\theta_2$  are anticommuting odd coordinates which satisfy the anticommutation relations

$$\theta_1^2 = \theta_2^2 = 0, \quad [\theta_1, \theta_2] = 0.$$

The  $\mathcal{D}_1 = \partial_{\theta_1} + \theta_1 \partial_x$  and  $\mathcal{D}_2 = \partial_{\theta_2} + \theta_2 \partial_t$  are two covariant derivatives which satisfy the anticommutation relations

$$\mathcal{D}_1^2 = \partial_x, \quad \mathcal{D}_2^2 = \partial_t, \quad [\mathcal{D}_1, \mathcal{D}_2] = 0.$$

The supersymmetric version of the sine-Gordon equation was introduced from purely physical motivations [43]. It is becoming increasingly interesting to investigate the supersymmetric sine-Gordon equation because of its close relation to string theories and statistical physics [44]-[46]. In recent publications, a superspace extension of the Lagrangian formulation has been established for the supersymmetric sine-Gordon equation [18]. The bilinear method is used to construct



multi-super soliton solutions [47]. The supersymmetric sine-Gordon equation admits a Lax pair, and a connection was established between its super-Backlund and super-Darboux transformations [48, 49]. The method of symmetry reduction is systematically applied in order to derive invariant solutions of the supersymmetric sine-Gordon equation [50]. The prolongation method of Wahlquist and Estabrook was used to find an infinite-dimensional superalgebra and the associated super Lax pairs [51].

Here we study the integrable properties of the supersymmetric sine-Gordon based on the use of generalized super Bell polynomials. The bilinear form, bilinear Backlund transformation, Lax pair and infinite conservation laws systematically are obtained with our method.

**Theorem 11.** Under the transformation

$$\Phi = 2i \ln(F/G),$$

the supersymmetric sine-Gordon equation (5.2) admits the bilinear form

$$2S_1S_2F \cdot F + G^2 = 0, \quad 2S_1S_2G \cdot G + F^2 = 0, \quad (5.3)$$

where  $F, G : \mathbb{R}_\Lambda^{2,2} \rightarrow \Lambda_0$  are two bosonic functions.

*Proof.* As before, the invariance of the supersymmetric sine-Gordon equation (5.2) under the scale transformation

$$x \rightarrow \lambda x, \quad t \rightarrow \lambda^{-1}t, \quad \theta_1 \rightarrow \lambda^{1/2}\theta_1, \quad \theta_2 \rightarrow \lambda^{-1/2}\theta_2, \quad \Phi \rightarrow \Phi$$

shows that the dimension of the bosonic superfield  $\Phi$  is zero, and so we may introduce a dimensionless bosonic field  $q$  by setting

$$\Phi = cq, \quad (5.4)$$

in which  $c \in \Lambda_0$  is free constant to be determined. Substituting (5.4) into (5.2) yields

$$2\mathcal{D}_1\mathcal{D}_2q = P_{\theta_1\theta_2}(p+q) - P_{\theta_1\theta_2}(p-q) = i(e^{-icq} - e^{icq})/c \quad (5.5)$$

where  $p : \mathbb{R}_\Lambda^{2,2} \rightarrow \Lambda_0$  is an auxiliary function. If one chooses the constant  $c = 2i$ , the equation (5.5) is then cast into a linear combination form of  $P$ -polynomials

$$2P_{\theta_1\theta_2}(p+q) - 2P_{\theta_1\theta_2}(p-q) + \exp(-2q) - \exp(2q) = 0,$$

which can be decoupled into a system

$$\begin{aligned} E_1(p, q) &= 2P_{\theta_1\theta_2}(p+q) + \exp(-2q) = 0, \\ E_2(p, q) &= 2P_{\theta_1\theta_2}(p-q) + \exp(2q) = 0. \end{aligned} \quad (5.6)$$

Multiplying the first equation by  $\exp(p+q)$ , the second equation by  $\exp(p-q)$  in the equation (5.6) yields

$$\begin{aligned} 2\exp(p+q)P_{\theta_1\theta_2}(p+q) + \exp(p-q) &= 0, \\ 2\exp(p-q)P_{\theta_1\theta_2}(p-q) + \exp(p+q) &= 0. \end{aligned} \quad (5.7)$$

By transformation

$$q = \ln(F/G), \quad p = \ln(FG) \iff \Phi = 2iq = 2i \ln(F/G), \quad p = \ln(FG)$$

and using the property (3.16), then the equation (5.7) gives the bilinear form (5.3) for the supersymmetric sine-Gordon equation (5.2).  $\square$

**Theorem 12.** Let  $(F, G)$  be a solution of the equation (5.3), then  $(\tilde{F}, \tilde{G})$  satisfying

$$\begin{aligned} S_1\tilde{G} \cdot G &= \lambda g\tilde{F}F, \quad S_1\tilde{F} \cdot F = -\lambda g\tilde{G}G, \\ S_2\tilde{F} \cdot G &= \frac{1}{4\lambda}g\tilde{G}F, \quad S_2\tilde{G} \cdot F = -\frac{1}{4\lambda}g\tilde{F}G, \\ \mathcal{D}_1g &= \lambda \left( \frac{F\tilde{F}}{G\tilde{G}} - \frac{G\tilde{G}}{F\tilde{F}} \right), \quad \mathcal{D}_2g = \frac{1}{4\lambda} \left( \frac{G\tilde{F}}{F\tilde{G}} - \frac{F\tilde{G}}{G\tilde{F}} \right). \end{aligned} \quad (5.8)$$

is another solution of the equation (5.3), where  $g : \mathbb{R}_\Lambda^{2,2} \rightarrow \Lambda_1$  fermionic auxiliary superfield and  $\lambda \in \Lambda_0$  is even parameter.

*Proof.* In order to obtain the bilinear Bäcklund transformation and Lax pairs of the equation (5.2), let  $p, q$  and  $\tilde{p}, \tilde{q}$  be two solutions of the equation (5.6) and consider the associated two-field condition

$$\begin{aligned} E_1(\tilde{p}, \tilde{q}) - E_1(p, q) &= 2\mathcal{D}_1\mathcal{D}_2(\tilde{p} - p) - 2\mathcal{D}_1\mathcal{D}_2(\tilde{q} - q) \\ &\quad + e^{\tilde{q}+q}(e^{\tilde{q}-q} - e^{q-\tilde{q}}) = 0, \\ E_2(\tilde{p}, \tilde{q}) - E_2(p, q) &= 2\mathcal{D}_1\mathcal{D}_2(\tilde{p} - p) + 2\mathcal{D}_1\mathcal{D}_2(\tilde{q} - q) \\ &\quad + e^{-(\tilde{q}+q)}(e^{\tilde{q}-q} - e^{q-\tilde{q}}) = 0, \end{aligned} \tag{5.9}$$

where

$$\tilde{p} = \ln(\tilde{F}\tilde{G}), \quad \tilde{q} = \ln(\tilde{F}/\tilde{G}),$$

We introduce variables

$$v_1 = \ln(\tilde{G}/G), \quad v_2 = \ln(\tilde{F}/F), \quad v_3 = \ln(\tilde{F}/G), \quad v_4 = \ln(\tilde{G}/F),$$

$$w_1 = \ln(\tilde{G}G), \quad w_2 = \ln(\tilde{F}F), \quad w_3 = \ln(\tilde{F}G), \quad w_4 = \ln(\tilde{G}F),$$

from which, we have relations

$$\tilde{q} - q = v_2 - v_1 = w_3 - w_4, \quad \tilde{q} + q = v_3 - v_4 = w_2 - w_1, \tag{5.10}$$

$$\tilde{p} - p = v_1 + v_2 = v_3 + v_4, \quad \tilde{p} + p = w_1 + w_2 = w_3 + w_4$$

and

$$v_1 = v_4 + q, \quad v_2 = v_3 - q, \quad w_1 = w_4 - q, \quad w_2 = w_3 + q. \tag{5.11}$$

By using the mixed variables (5.10), it follows that from (5.9)

$$4\mathcal{D}_1\mathcal{D}_2v_1 + e^{v_3-v_4}(e^{v_2-v_1} - e^{v_1-v_2}) = 0, \tag{5.12}$$

$$4\mathcal{D}_1\mathcal{D}_2v_2 + e^{v_4-v_3}(e^{v_1-v_2} - e^{v_2-v_1}) = 0,$$

which may produce the required bilinear Bäcklund transformation under an appropriate additional constraint. We choose a constraint

$$\mathcal{D}_1v_1 = \mathcal{Y}_{\theta_1}(v_1) = \lambda g e^{v_3-v_4}, \tag{5.13}$$

where  $g : \mathbb{R}_\Lambda^{2,2} \rightarrow \Lambda_1$  fermionic auxiliary superfield and  $\lambda \in \Lambda_0$  is even parameter. The fermionic function  $g$  is introduced because of supersymmetry and the oddness of the superspace derivatives  $\mathcal{D}_1, \mathcal{D}_2$ . The constraint (5.13) reduces the first equation in (5.12) into

$$-4\lambda\mathcal{D}_2g - 4\lambda g\mathcal{D}_2(v_3 - v_4) + e^{v_2-v_1} - e^{v_1-v_2} = 0. \quad (5.14)$$

Since the term  $\mathcal{D}_2(v_3 - v_4)$  should be fermionic function, we make a constraint

$$\mathcal{D}_2(v_3 - v_4) = \frac{1}{4\lambda}gh, \quad (5.15)$$

where  $h$  is a bosonic function to be determined. On account of this constraint, it follows from (5.14) that

$$\mathcal{D}_2g = \frac{1}{4\lambda}(e^{v_2-v_1} - e^{v_1-v_2}), \quad (5.16)$$

which holds because  $g^2 = 0$ ,  $g$  being fermionic.

By means of the system (5.15) and (5.16), the second equation in (5.12) reads

$$\mathcal{D}_2(\mathcal{D}_1v_2 + \lambda ge^{v_4-v_3}) = 0,$$

which is satisfied if we choose

$$\mathcal{Y}_{\theta_1}(v_2) = \mathcal{D}_1v_2 = -\lambda ge^{v_4-v_3}. \quad (5.17)$$

On the one hand, using the relation (5.11), we have

$$\begin{aligned} 4\mathcal{D}_1\mathcal{D}_2(v_3 - v_4) + 4\mathcal{D}_2\mathcal{D}_1(v_1 - v_2) &= 8\mathcal{D}_1\mathcal{D}_2g \\ &= (e^{v_2-v_1} - e^{v_1-v_2})(e^{v_3-v_4} + e^{v_4-v_3}) + (e^{v_2-v_1} + e^{v_1-v_2})(e^{v_3-v_4} - e^{v_4-v_3}). \end{aligned} \quad (5.18)$$

On the other hand, it follows from (5.13), (5.16) and (5.17) that

$$\begin{aligned} 4\mathcal{D}_2\mathcal{D}_1(v_1 - v_2) &= 4\lambda(\mathcal{D}_2g)(e^{v_3-v_4} + e^{v_4-v_3}) \\ &= (e^{v_2-v_1} - e^{v_1-v_2})(e^{v_3-v_4} + e^{v_4-v_3}). \end{aligned} \quad (5.19)$$

Combining (5.15), (5.18) and (5.19) yields

$$4\mathcal{D}_2\mathcal{D}_1(v_3 - v_4) = \frac{1}{\lambda}(\mathcal{D}_1g)h = (e^{v_3-v_4} - e^{v_4-v_3})(e^{v_2-v_1} + e^{v_1-v_2}),$$

which implies that we may choose

$$\mathcal{D}_1g = \lambda(e^{v_3-v_4} - e^{v_4-v_3}) \quad (5.20)$$

and

$$h = (e^{v_2-v_1} + e^{v_1-v_2}).$$

Thus, we have

$$\mathcal{D}_2(v_3 - v_4) = \frac{1}{4\lambda}g(e^{v_2-v_1} + e^{v_1-v_2}),$$

which can be written as a pair of  $\mathcal{Y}$ -polynomials

$$\mathcal{Y}_{\theta_2}(v_3) = \mathcal{D}_2v_3 = \frac{1}{4\lambda}ge^{v_1-v_2}, \quad \mathcal{Y}_{\theta_2}(v_4) = \mathcal{D}_2v_4 = -\frac{1}{4\lambda}ge^{v_2-v_1}, \quad (5.21)$$

Combining (5.13), (5.16), (5.17), (5.20) and (5.21) gives bilinear Bäcklund transformation (5.8) of the supersymmetric sine-Gordon equation.  $\square$

Finally we derive Lax pair of the supersymmetric sine-Gordon equation.

**Theorem 13.** The supersymmetric sine-Gordon equation (5.2) admits a Lax pair

$$\begin{aligned} \mathcal{D}_1\Psi &= M\Psi = \begin{pmatrix} -\frac{1}{2}i\mathcal{D}_1\Phi & \lambda g \\ \lambda g & \frac{1}{2}i\mathcal{D}_1\Phi \end{pmatrix} \Psi, \\ \mathcal{D}_2\Psi &= N\Psi = \begin{pmatrix} 0 & -\frac{1}{4\lambda}ge^{-i\Phi} \\ -\frac{1}{4\lambda}ge^{i\Phi} & 0 \end{pmatrix} \Psi, \end{aligned} \quad (5.22)$$

together with

$$\mathcal{D}_1g = \lambda \left( \frac{\psi_3}{\psi_4} - \frac{\psi_4}{\psi_3} \right), \quad \mathcal{D}_2g = \frac{1}{\lambda} \left( e^{i\Phi} \frac{\psi_3}{\psi_4} - e^{-i\Phi} \frac{\psi_4}{\psi_3} \right),$$

where  $\Psi = (\psi_3, \psi_4)^T$ .

Making use of the Hopf-Cole transformation

$$v_3 = \ln \psi_3, \quad v_4 = \ln \psi_4,$$

then the system (5.13), (5.16), (5.17), (5.20) and (5.21) can be linearized into a Lax pair (5.22). It is easy to check that the integrability condition

$$\mathcal{D}_2 M + \mathcal{D}_1 N - [M, N] = 0$$

is satisfied if  $\Phi$  is a solution of the sine-Gordon equation (5.2).

If we choose a transformation

$$\phi_1 = \psi_4^2, \quad \phi_2 = \psi_3^3, \quad g = \frac{\phi_3}{2i\psi_3\psi_4}$$

then the Lax pair (5.22) is also equivalent to a linear system in  $3 \times 3$  matrix form

$$\begin{aligned} \mathcal{D}_1 \Omega &= \frac{1}{4} \begin{pmatrix} 4\mathcal{D}_1 \Phi & 0 & \lambda \\ 0 & -4\mathcal{D}_1 \Phi & -\lambda \\ -4\lambda & 4\lambda & 0 \end{pmatrix} \Omega, \\ \mathcal{D}_2 \Omega &= \frac{1}{16\lambda} \begin{pmatrix} 0 & 0 & e^{i\Phi} \\ 0 & 0 & -e^{-i\Phi} \\ 4e^{-i\Phi} & -4e^{-i\Phi} & 0 \end{pmatrix} \Omega, \end{aligned} \tag{5.23}$$

where  $\Omega = (\phi_1, \phi_2, \phi_3)^T$ ,  $\phi_1, \phi_2 : \mathbb{R}_\Lambda^{2,2} \rightarrow \Lambda_0$  are bosonic functions and  $\phi_3 : \mathbb{R}_\Lambda^{2,2} \rightarrow \Lambda_1$  is a fermionic function. The system (5.23) also can be obtained from (5.13), (5.16), (5.17), (5.20) and (5.21) by setting

$$2(v_4 - v_3) = \ln \frac{\phi_1}{\phi_2}, \quad 2(v_2 - v_1) = -2i\Phi + \ln \frac{\phi_1}{\phi_2}, \quad g = \frac{\phi_3}{2i\sqrt{\phi_1\phi_2}}.$$

The compatibility of the linear system (5.23) in superspace is equivalent to the equation (5.2). The system (5.23) is the same as obtained in [41], but here it is derived systematically from the super Bell polynomials and Lax pairs.  $\square$

Noting the transformation relation

$$v_1 - v_2 = i(\Phi - \tilde{\Phi})/2, \quad v_3 - v_4 = -i(\Phi + \tilde{\Phi})/2,$$

then it follows from equations (5.13), (5.17), (5.18), (5.21) and (5.22) that

$$\begin{aligned}\mathcal{D}_1(\Phi - \tilde{\Phi}) &= \lambda g \cos\left(\frac{\Phi + \tilde{\Phi}}{2}\right), \\ \mathcal{D}_2(\Phi + \tilde{\Phi}) &= \frac{1}{4\lambda} g \cos\left(\frac{\Phi - \tilde{\Phi}}{2}\right), \\ \mathcal{D}_1 g &= \lambda \sin\left(\frac{\Phi + \tilde{\Phi}}{2}\right), \quad \mathcal{D}_2 g = \frac{1}{4\lambda} \sin\left(\frac{\Phi - \tilde{\Phi}}{2}\right),\end{aligned}\tag{5.24}$$

which is the Bäcklund transformation of the supersymmetric sine-Gordon equation. The compatibility of the Bäcklund transformation (5.24) is the supersymmetric sine-Gordon equation for both  $\Phi$  and  $\tilde{\Phi}$  separately. The super Bäcklund transformation (5.24) reduces to the classical Bäcklund transformation of the purely bosonic sine-Gordon equation when fermions are equal to zero.

## 6. Concluding Remarks

In this paper, we have introduced a class of super Bell polynomials which play an important role in the characterization of bilinear Bäcklund transformation, Lax pairs and infinite conservation laws of supersymmetric equations. To the knowledge of the authors, this is the first work on the super Bell polynomials and their applications to super integrable systems. We believe that there are still many interesting deep relations between generalized Bell polynomials and integrable structures, which remain open and worth to be considered. For instance, (i) How to explore the relations between the super Bell polynomials with symmetries, Hamiltonian functions, etc. (ii) How to define a class of discrete Bell polynomials and apply them in discrete equations. We have some ideas on these questions

and will intend to return to them in some future publications.

## Acknowledgment

The work described in this paper was supported by grants from the CityU (Project No. 7002440), the National Science Foundation of China (No. 10971031) and Shanghai Shuguang Tracking Project (No. 08GG01).

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